FINITE ENERGY SUM RULES FOR THE CROSS SECTION OF e^+e^--ANNIHILATION INTO HADRONS IN QCD

K.G. CHETYRKIN, N.V. KRASNIKOV and A.N. TAVKHELIDZE
The Institute for Nuclear Research of the Academy of Sciences of the USSR, Moscow 117312, USSR

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Finite energy sum rules (FESR) for the cross section of e^+e^--annihilation into hadrons are obtained within the framework of QCD. The FESR for the standard 4-quark model are in satisfactory agreement with experiment. The best description of the data is given by the 5-quark model, with the electric charge of the 5th quark $Q_5 = -1/3$.

Currently one of the most popular models of strong interactions is quantum chromodynamics (QCD), i.e. the gauge theory of colored quarks and gluons. However, due to the lack of clear understanding of the "confinement puzzle" QCD is not directly applicable to the calculations of the cross sections of physical processes on mass-shell even at high energies. For instance as applied to the process of e^+e^--annihilation into hadrons QCD gives information about the behaviour of the hadron vacuum polarization function $\Pi(z)$ only far enough from the cut $[4m_\pi^2, \infty)$ at which the imaginary part of $\Pi(z)$ is proportional to the quantity $R(S) = \sigma(e^+e^- - \text{hadrons})/\sigma(e^+e^- - \mu^+\mu^-)$ [1–4].

Let $\Pi_{\text{th}}(z)$ be the value of $\Pi(z)$ evaluated in QCD up to Nth order in the quark–gluon coupling constant α_s. In ref. [4] convincing arguments were given in favour of the validity of the approximate equality

$$\Pi(z) \approx \Pi_{\text{th}}(z), \qquad \operatorname{Im} z \geqslant \Delta, \tag{1}$$

where Δ is a parameter dependent on N.

In this paper we would like to call attention to the obvious fact that the analytical properties of $\Pi(z)$ expressed by the Kallen–Lehman representation

$$\Pi(z) = z \int\limits_{4m_\pi^2}^{\infty} \frac{R(s')\,\alpha s'}{s'(s'-z)}, \tag{2}$$

and eq. (1) involve a host of integral sum rules connecting $R(s)$ and

$$R_{\text{th}}(s) = \frac{1}{2\pi i}\left(\Pi_{\text{th}}(s+i\epsilon) - \Pi_{\text{th}}(s-i\epsilon)\right).$$

Indeed, integrating the function $\Pi(z)z^n$, $n = 0, +1, +2, \ldots$, over the contour shown in fig. 1a and the function $\Pi(z)/z^k$, $k = 0, \pm 1, \pm 2, \ldots$, over the contour of fig. 1b and using, together with equalities (1) and (2), the Cauchy theorem, we obtain finite energy sum rules (FESR)

$$\int\limits_{4m_\pi^2}^{s} R(s')\,s'^n\,\mathrm{d}s' \approx \int\limits_0^s R_{\text{th}}(s')\,s'^n\,\mathrm{d}s', \tag{3}$$

$$\int\limits_{s_0}^{s} R(s')/s'^k\,\mathrm{d}s' \approx \int\limits_{s_0}^{s} R_{\text{th}}(s')/s^k\,\mathrm{d}s'. \tag{4}$$

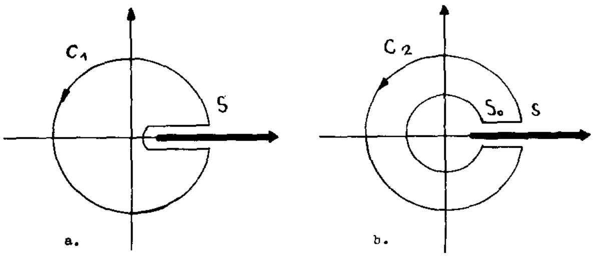

Fig. 1. The integration path used in the derivation of the FESR (3) and (4).

Notice that the error due to invalidity of perturbation theory for $\mathrm{Im}\, z < \Delta$ is equal in order of magnitude to $\Delta/(\pi s)$ for the FESR (3) and $\Delta/[\pi(s - s_0)]$ for the FESR (4) (to within possible logarithmic factors).

The sum rules (3) were discussed in refs. [5,6] on the basis of analogy with duality sum rules for strong interactions [7]. Those works, however, contained no convincing arguments in favour of a particular choice of $R_{\text{th}}(s)$. In quantum chromodynamics the FESR (3) at $n = 0$ was used in refs. [8,9] to find the parameter defining the scale of the quark–gluon coupling constant α_s from experimental data for $s \leqslant 1\ \text{GeV}^2$.

We compare the FESR (4) with experiment at $s_0 = 9\ \text{GeV}^2$, $s = 60\ \text{GeV}^2$ for the values of k variable within $-10 \leqslant k \leqslant 10$. For R_{th} we used the formula [4]

$$R_{\text{th}}(s) = \tfrac{3}{2} \sum_i Q_i^2\, (3 - v_i^2)\, v_i (1 + \tfrac{4}{3}\, \alpha_s f(v_i)), \tag{5}$$

$$v_i = \left(1 - \frac{4m_i^2}{s}\right)^{1/2}, \quad f(v) = \frac{\pi}{2v} - \frac{(3+v)}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right),$$

$$\alpha_s = 12\pi \left[33 \ln\left(\frac{s}{\Lambda^2}\right) - 2 \sum_i \ln\left(\frac{s + 5m_i^2}{\Lambda^2 + 5m_i^2}\right)\right]^{-1},$$

where m_i, Q_i are the mass and charge of the ith flavour quark. (We note that in eq. (5) the quark masses correspond to the position of the poles of quark propagators; for the discussion of other possible ways of introducing the parameters corresponding to quark masses in perturbation theory, see ref. [9].) The masses of light u, d and s quarks were taken to be zero and for the quantity Λ we used the value $\Lambda = 500$ meV which is preferable as regards the description of deep inelastic lepton–hadron reactions [10] and the mean value of $R(s)$ at $s \leqslant 1\ \text{GeV}^2$ in terms of quantum chromodynamics. To calculate the integrals with $R(s)$, we used an analytical fit for R, obtained by combining the formulas of refs. [11,12] at $\sqrt{s} \leqslant 5$ GeV, and for $\sqrt{s} \geqslant 5$ GeV we put $R = 5$ [13]. In this case for R_{th} we used the expression (5) with the addition corresponding to production of a pair of heavy leptons $\tau^+\tau^-$ [14]. For the standard four-quark model with $m_c = 1.25\ \text{GeV}/c^2$ [15] the FESR (4) holds to within 20%(20%)[‡1]. For the five-quark model with $Q_5 = -1/3$ the best agreement between theory and experiment is achieved with masses of the fourth and fifth quarks $m_4 = m_5 = 1.6\ (\text{GeV}/c)^2$, the FESR (4) holding to within 10%(12%). However, it should be noted that the values m_4 and m_5 are very sensitive to the particular choice of Λ. So, if we take $\Lambda = 700$ meV, we should get $m_4 = 1.6\ \text{GeV}/c^2$, $m_5 = 2.3\ \text{GeV}/c^2$.

If we assume that the contribution of charmed and usual hadrons to R are additive and confine ourselves to the zero-order approximation of perturbation theory, then it follows from eq. (4) that in the standard model ($k \geqslant 2$)

$$\int_{4m_\psi^2}^{\infty} \frac{R^c(s')}{s'^k}\, ds' \approx \int_{4m_c^2}^{\infty} \frac{R^c_{\text{th}}(s')}{s'^k}\, ds', \tag{6}$$

where R^c is the contribution to R from production of hadrons containing c-quarks.

The FESR (6) were first obtained in ref. [15] by calculating $(d^n\Pi^c/ds^n)\,(0)$. A comparison of eq. (6) with experiment made in that work showed that the FESR (6) are in reasonable agreement with experiment only when $k < 7$. We thus come to the conclusion that rejection of the hypothesis of additivity and inclusion of the interaction substantially improve the agreement between the FESR and experiment.

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‡1 Given here and below in brackets are the relative values of errors in cases where the interaction is disregarded ($\alpha_s \equiv 0$).

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